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On the origin of meandering and braiding in alluvial streams

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The paper describes a hydrodynamic stability analysis of the flow in an alluvial channel in which dunes have developed along the bed. The purpose is to develop a mathematical model describing the three-dimensional flow leading to instability of an originally straight channel. The model offers an explanation of the fact that some channels tend to meander, others to braid.

1. Introduction

The particular river-morphology problem known as meandering has been studied by several scientists and from many different points of view, as will emerge from the short survey given in this introduction.

The first quantitative description, today known as the regime theory, was initiated mainly by British engineers on the basis of an extensive collection of data from channels in India. In the original form, the regime equations related the meander length L to the formative discharge Q only,

$$L \sim \sqrt{Q},$$

but later versions have modified this simple relation to include some dependence on the sediment properties, see for instance Blench (1966).

As a supplement to the field observations, laboratory investigations have been used intensively in recent years and have contributed considerably to the understanding of river mechanics (Leopold & Wolman 1957; Ackers & Charlton 1970).

On the theoretical side several attempts have been made to understand the basic mechanism of meandering. Most attempts have been purely deterministic, but attention should be paid to an interesting statistical approach introduced by Langbein & Leopold (1966) in the so-called theory of minimum variance.

The possibility of describing the origin of meandering as a problem of instability was investigated in detail by Callander (1969), using a two-dimensional flow model. This theory neglects the velocity variation along verticals and includes the internal friction by a one-dimensional description, corresponding to gradually varying flow. Nevertheless, the results are in rather good agreement with experiments.

Engelund & Hansen (1967) suggested that the meander length is determined by the hydraulic resistance (friction factor) and the depth. They also demonstrated that this does not contradict the regime theory, although one would probably expect this on a superficial consideration.

Finally, it should be mentioned that Einstein & Shen (1964) succeeded in explaining important features of meandering in straight alluvial channels by considering secondary currents induced by the difference in shear stress at the two banks or by the asymmetry of the channel cross-section. This work was later continued by Shen & Komura (1968).

In the present paper an attempt has been made to develop a theory taking account of the three-dimensional character of the flow. Not much work has been done on three-dimensional flow in channels, but reference may be made to a paper by Reynolds (1965) considering the formation of dunes. This work was later continued by Engelund & Fredsøe (1971).

2. The basic equations

As in every hydrodynamic stability analysis we consider a basic (undisturbed) flow on which a perturbation is superposed. For the basic system we consider a steady and uniform flow in a channel with rectangular cross-section. The side walls are fixed, while the bed consists of non-cohesive sediment. The channel width is B and the water depth D, as indicated in the definition sketch, figure 1. We introduce a co-ordinate system with the x_1 axis in the flow direction, x_2 horizontal in the water surface and x_3 perpendicular to the bed.

If the width B is sufficiently large compared with the depth D (which is usually the case in natural streams), the direct effect of the side walls is restricted to rather narrow regions. In order to facilitate the analysis the side-wall friction has been neglected in the following. In these circumstances a two-dimensional turbulent flow has a mean velocity profile given by the defect law

$$(U_0 - U)/U_{f0} = f(x_3/D), \tag{1}$$

where U_0 is the surface velocity, U the velocity at a distance x_3 below the surface, and U_{f0} is the friction or shear velocity, defined as

$$U_{f0} = (\tau_0/\rho)^{\frac{1}{2}}, \tag{2}$$

where τ_0 denotes the bed shear stress and ρ the fluid density.

For the present analysis, it is convenient to assume a velocity distribution of the following form, putting $\xi = x_3/D$:

$$U = U_0 \cos \beta\xi, \tag{3}$$

or

$$\frac{U_0 - U}{U_{f0}} = \frac{U_0}{U_{f0}}[1 - \cos \beta\xi] \approx \frac{U_0}{U_{f0}} \tfrac{1}{2}\beta^2\xi^2.$$

As the right-hand side is to be a universal function of ξ, according to (1), and hence must contain no parameters, it is seen that β must have the following form:

$$\beta = [14 U_{f0}/U_0]^{\frac{1}{2}}, \tag{4}$$

the factor 14 being selected to obtain the best fit for the actual velocity profile.

Equation (3) gives a satisfactory description of the velocity profile in the upper part of the flow, but fails to describe the abrupt decrease of velocity taking place close to the bottom. Instead, (3) indicates a finite velocity U_{b0} at the bottom:

$$U_{b0} = U_0 \cos \beta. \tag{5}$$

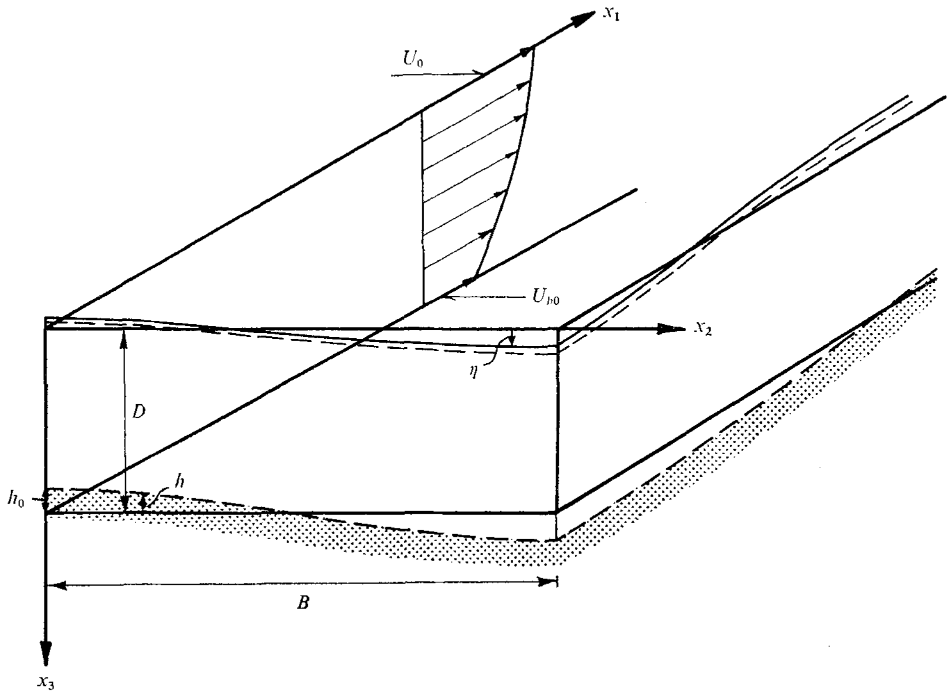

FIGURE 1. Definition sketch.

It should be mentioned that the distribution given by (3) is very close to the parabolic velocity distribution obtained by assuming a constant eddy viscosity ϵ (Engelund 1970, p. 227).

In order to investigate the stability of the described uniform flow, we consider next the flow taking place on a perturbed bed. This means that the plane bed is deformed to a position h above its original position as indicated by the dotted curves in figure 1. The character of meandering requires a double-periodic disturbance, so that h is given by

$$h = h_0 \cos k_2 x_2 \exp(ik_1 x_1), \tag{6}$$

in which h_0 is the amplitude, and k_1 and k_2 are the cyclic wavenumbers. In a first-order theory second and higher powers in h_0 are neglected.

The perturbed flow may now be described mathematically by the equation of motion

$$\frac{\partial v_i}{\partial x_j} v_j = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + g_i - \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j} \tag{7}$$

using Cartesian tensor notation. In (7) v_i is the velocity vector, p the fluid pressure, g_i the acceleration of gravity and τ_{ij} the deviatoric stress tensor. The complete velocity vector v_i is now written as the sum of the basic velocity U_i and the perturbation velocity u_i:

$$v_i = U_i + u_i.$$

For the basic flow we find on the assumptions stated above that

$$U_1 = U(x_3); \quad U_2 = U_3 = 0.$$

Again, in a linear theory u_i is small compared with U, so that second- and higher-order terms may be neglected. In the deviatoric stress only the shear stresses in the horizontal plane are retained. The complete expression is

$$\frac{1}{\rho}\tau_{ij} = -\epsilon\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right).$$

In the circumstances considered, where only very large wave periods are investigated, it is further justifiable to regard the deviations in the x_1 and x_2 directions as small quantities when compared with the derivatives with respect to x_3.

For the gravity term we substitute the expression

$$g_i = -\partial(gz)/\partial x_i,$$

where z is the level (geometric head). Then, the expanded form of (7) becomes

$$\frac{\partial}{\partial x_1}\left(\frac{p}{\rho}+gz\right) = -U\frac{\partial u_1}{\partial x_1}-u_3U'+\epsilon\left(U''+\frac{\partial^2 u_1}{\partial x_3^2}\right), \tag{7a}$$

$$\frac{\partial}{\partial x_2}\left(\frac{p}{\rho}+gz\right) = -U\frac{\partial u_2}{\partial x_1}+\epsilon\frac{\partial^2 u_2}{\partial x_3^2}, \tag{7b}$$

$$\frac{\partial}{\partial x_3}\left(\frac{p}{\rho}+gz\right) = -U\frac{\partial u_3}{\partial x_1}, \tag{7c}$$

where primes denote differentiation with respect to x_3. To eliminate the non-periodic part we write

$$(p/\rho)+gz = g(z_0+D)+P, \tag{8}$$

the index 0 referring to the undisturbed bed level. Then P is periodic and

$$\frac{\partial}{\partial x_1}(gz_0+gD) = \frac{\partial}{\partial x_1}(gz_0) = -gS_0,$$

where S_0 is the mean channel slope. For the uniform basic flow the following relation will hold:

$$\epsilon U'' = -gS_0,$$

so that equations (7*a*, *b*, *c*) are reduced to

$$\frac{\partial P}{\partial x_1} = -U\frac{\partial u_1}{\partial x_1}-u_3U'+\epsilon\frac{\partial^2 u_1}{\partial x_3^2}, \tag{9a}$$

$$\frac{\partial P}{\partial x_2} = -U\frac{\partial u_2}{\partial x_1}+\epsilon\frac{\partial^2 u_2}{\partial x_3^2}, \tag{9b}$$

$$\frac{\partial P}{\partial x_3} = -U\frac{\partial u_3}{\partial x_1}. \tag{9c}$$

Further, the condition of incompressibility is expressed by the equation of continuity

$$\partial u_i/\partial x_i = 0.$$

The assumption of periodicity is now introduced by the following expressions, in which $\xi = x_3/D$:

$$P = U_{b0}^2\phi(\xi)\cos k_2x_2 \exp(ik_1x_1), \tag{10a}$$

$$u_1 = U_{b0}f_1'(\xi)\cos k_2x_2 \exp(ik_1x_1), \tag{10b}$$

$$u_2 = iU_{b0}f_2'(\xi)\sin k_2x_2 \exp(ik_1x_1), \tag{10c}$$

$$u_3 = iU_{b0}f_3(\xi)\cos k_2x_2 \exp(ik_1x_1). \tag{10d}$$

ϕ and f_i are unknown complex functions of ξ. The continuity equation makes it possible to eliminate one of these unknown functions, say f_1:

$$f_1' = -\frac{k_2}{k_1}f_2 - \frac{1}{k_1D}f_3. \tag{11}$$

When the expressions (10) are inserted in equations (9), we end up with the following basic equations:

$$\phi = -\frac{U}{U_{b0}}f_1' - \frac{U'}{k_1DU_{b0}}f_3 - i\frac{\epsilon}{k_1D^2U_{b0}}f_1''', \tag{12a}$$

$$\phi = -\frac{U}{U_{b0}}\frac{k_1}{k_2}f_2' - i\frac{\epsilon}{k_2D^2U_{b0}}f_2''', \tag{12b}$$

$$\phi' = \frac{U}{U_{b0}}k_1Df_3, \tag{12c}$$

the prime now indicating differentiation with respect to ξ.

The immediate problem is now to determine the unknown functions from (11) and (12). In order to derive an equation for only one of the unknown functions the following procedure has been applied: f_2 is found from (11) and inserted in (12*b*). Elimination of ϕ between (12*a*) and (12*b*) then gives a relationship between f_1 and f_3. This equation is differentiated with respect to ξ. Then (12*a*) is differentiated with respect to ξ, and ϕ' is eliminated using (12*c*). From the resulting two equations the following differential equation in f_3 is obtained:

$$\frac{i\epsilon}{k_1D^2U}f_3^{\text{iv}} + f_3'' = \left\{\frac{U''}{U} + (k_1^2+k_2^2)D^2\right\}f_3. \tag{13}$$

With f_3 known we may obtain f_2 from (12*b*) and (12*c*):

$$\frac{d}{d\xi}\left[\frac{U}{U_{b0}}f_2' + \frac{i\epsilon}{k_1D^2U_{b0}}f_2'''\right] = -\frac{U}{U_{b0}}k_2Df_3,$$

or

$$\frac{U}{U_{b0}}f_2' + \frac{i\epsilon}{k_1D^2U_{b0}}f_2''' = -\int_0^\xi \frac{U}{U_{b0}}k_2Df_3\,d\xi - \frac{k_2}{k_1}\phi(0). \tag{14}$$

Finally, f_1 is found from (11).

For later use we notice that the following expression for ϕ may be obtained from (12*a*), (12*b*) and (11):

$$\left[1+\frac{k_2^2}{k_1^2}\right]\phi = -\frac{U'}{k_1DU_{b0}}f_3 + \frac{1}{k_1D}\left[\frac{U}{U_{b0}}f_3' + \frac{i\epsilon}{k_1D^2U_{b0}}f_3'''\right]. \tag{15}$$

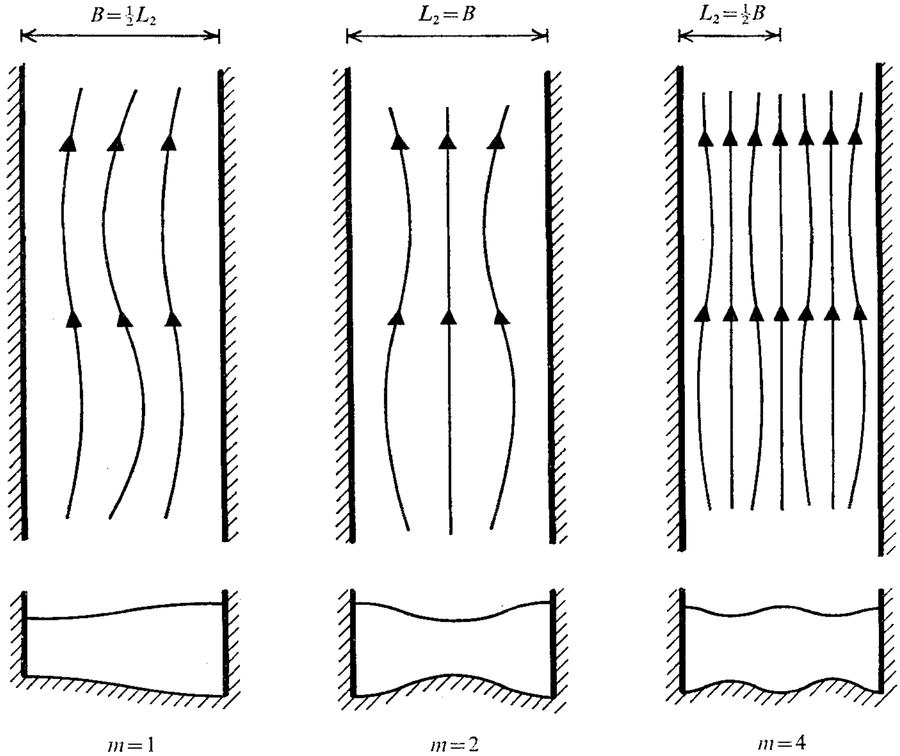

FIGURE 2. Sketch illustrating meandering ($m = 1$) and braiding ($m = 2$ and 4). Typical streamline patterns (above) and cross-sections (below).

3. The boundary conditions

The first requirement concerns the impermeability of the side walls expressed by $u_2 = 0$ for $x_2 = 0$ and $x_2 = B$. From (10*c*) we find that this condition is fulfilled if

$$k_2 B = m\pi \quad \text{or} \quad mL_2 = 2B, \tag{16}$$

where L_2 is the wavelength and m is a positive integer; m equal to unity corresponds to ordinary meandering, while m greater than unity describes a braiding stream, see figure 2.

Along the bed we have the kinematical boundary condition

$$u_3 = -U(\partial h/\partial x_1), \quad \xi = 1 - h/D,$$

which expresses the impermeability of the river bed. After insertion of (6) and (10*d*), this reduces to

$$f_3(1) = -h_0 k_1 \tag{17}$$

as a linear theory permits the undisturbed bed level ($\xi = 1$) to be used instead of the actual one, $\xi = 1 - h/D$.

Along the free water surface the following conditions must be fulfilled.

(i) The pressure p must be zero, and the water surface consist of streamlines.

(ii) The two shear stress components τ_{13} and τ_{23} must vanish.

From (i) we conclude that the velocity vector must be perpendicular to the pressure gradient, which is expressed by the equation

$$v_i(\partial p/\partial x_i) = 0, \quad \xi = \eta, \tag{18}$$

where η denotes the deviation of the actual water surface from the undisturbed level, see figure 1. When the pressure gradient is evaluated from (8) and we put $v_i = U_i + u_i$, this boundary condition becomes

$$\frac{1}{g}\frac{\partial P}{\partial x_1} = -\frac{u_3}{U_0},$$

or, when (10*a*) and (10*d*) are inserted,

$$f_3(0) = -(k_1 U_0 U_{b0}/g)\,\phi(0) \tag{19}$$

as a linear theory permits the undisturbed level ($\xi = 0$) to be used instead of the actual one, $\xi = \eta$.

The condition of vanishing pressure is expressed by

$$-g\eta = P(\xi = \eta), \tag{20}$$

which is used below.

Next, we consider the shear stresses along the surface. It is seen that the transverse component τ_{23} vanishes if

$$\partial v_2/\partial x_3 = 0, \quad \xi = \eta,$$

or, from (10*c*),

$$f_2''(0) = 0. \tag{21}$$

As far as the longitudinal component τ_{13} is concerned, the derivation proceeds in a similar way:

$$\partial(U + u_1)/\partial\xi = 0, \quad \xi = \eta, \tag{22}$$

but there is the minor difference that the curvature of the unperturbed velocity profile introduces an additional linear term. Applying a truncated Taylor series we get

$$[U']_{\xi=\eta} = U'(0) + U''(0)\,\eta/D = U''(0)\,\eta/D.$$

By application of this and of (10*a*), (10*b*), (11), (19), (20) and (21) condition (22) may be rewritten as

$$f_3''(0) + \beta^2 f_3(0) = 0. \tag{23}$$

Finally, two boundary conditions are related to the hydraulic resistance along the river bed. Since the bed is composed of erodible alluvial sediment, a major contribution to the roughness comes from the formation of dunes and will consequently vary spatially, when the flow conditions change. In the present context these dunes are regarded as roughness elements.

Hence, what we need now is a relation describing the behaviour of alluvial streams in terms of the flow parameters. Here, it is assumed that the longitudinal hydraulic resistance can be calculated according to Engelund's similarity hypothesis (Engelund 1967; Engelund & Hansen 1967):

$$\theta' = 0{\cdot}06 + 0{\cdot}4\theta^2 \quad \text{on} \quad \xi = 1 - h/D, \tag{24}$$

where

$$\theta' = SD'/(s-1)\,d \tag{25}$$

and

$$\theta = \tau_{13}/\rho g(s-1)\,d, \tag{26}$$

in which s = relative density of the sediment, d = representative grain diameter (the mean fall diameter is used in the following) and S = actual energy gradient. D' is calculated from

$$V/(gD'S)^{\frac{1}{2}} = 6 + 2{\cdot}5 \ln (D'/2{\cdot}5d). \tag{27}$$

For the uniform basic flow we get from (24) and (27)

$$\frac{D'_0}{D} = \frac{0{\cdot}06(s-1)}{S_0 D/d} + \frac{0{\cdot}4 S_0 D/d}{s-1} \tag{28}$$

and

$$\frac{V_0}{U_{f0}} = \left\{6 + 2{\cdot}5 \ln \left(0{\cdot}4 \frac{D'_0}{D}\frac{D}{d}\right)\right\} \left(\frac{D'_0}{D}\right)^{\frac{1}{2}}, \tag{29}$$

which for given values of the independent parameters (D/d, S_0 and s) determines the dependent parameters V_0/U_{f0} and the Froude number $F_0 = S_0^{\frac{1}{2}} V_0/U_{f0}$.

For the perturbed terms in (24) and (27) we get after some manipulation (linear theory)

$$c_1 f_3''(1) + c_2 f_3(1) + c_3 f_3(0) = c_4 - c_1 k_2 D f_2''(1) - c_2 k_2 D\{f_2(1) - f_2(0)\}, \tag{30}$$

where

$$c_0 \equiv \frac{(V_0/U_{f0})\, D'_0/D}{2{\cdot}5(D'_0/D)^{\frac{1}{2}} + \frac{1}{2}(V_0/U_{f0})}, \quad c_1 \equiv -\frac{\epsilon U_{b0}}{D U_{f0}^2}\left\{\frac{0{\cdot}8 S_0 D/d}{s-1} + \tfrac{1}{2}c_0 - \frac{D'_0}{D}\right\},$$

$$c_2 \equiv -\beta c_0 \cot \beta, \quad c_3 \equiv ([D'_0/D] - \tfrac{1}{2}c_0) \cos \beta,$$

$$c_4 \equiv k_1 h_0 \left\{c_0 \beta \cot \beta - S_0 \frac{D}{d} \frac{0{\cdot}8}{(s-1)}\right\}.$$

Finally, we assume that the shear stress of the bed has the same direction as the velocity vector:

$$\frac{\tau_{23}}{\rho} = -\epsilon \frac{\partial u_2}{\partial x_3} = U_{f0}^2 \frac{u_2}{U_{b0}} \quad \text{for} \quad \xi = 1 - \frac{h}{D}, \tag{31}$$

which gives the condition

$$f_2''(1) + \frac{D U_{f0}^2}{\epsilon U_{b0}} f_2'(1) = 0. \tag{32}$$

In the following calculation we have used

$$\epsilon = 0{\cdot}077 D U_{f0}. \tag{33}$$

4. The numerical approach

The two ordinary differential equations (13) and (14) and the necessary two-point boundary conditions (17), (19), (21), (23), (30) and (32) are solved by mixed orthogonal collocation. The basic procedure is described by Villadsen & Stewart (1967). Among the virtues of the method of orthogonal collocation are its great generality and ease of application. It is relatively easy to set up the equations, to solve them and to vary the order of convergence.

The basic idea of orthogonal collocation is that the solution of the differential equation is represented as a finite series of orthogonal polynomials, and the unknown coefficients in this representation are found by satisfying the associated conditions and the differential equation at an appropriate number of selected points (the collocation points). The zeros of the shifted Legendre polynomials have been used as the collocation points. The integration of f'_2 in (30) is performed using

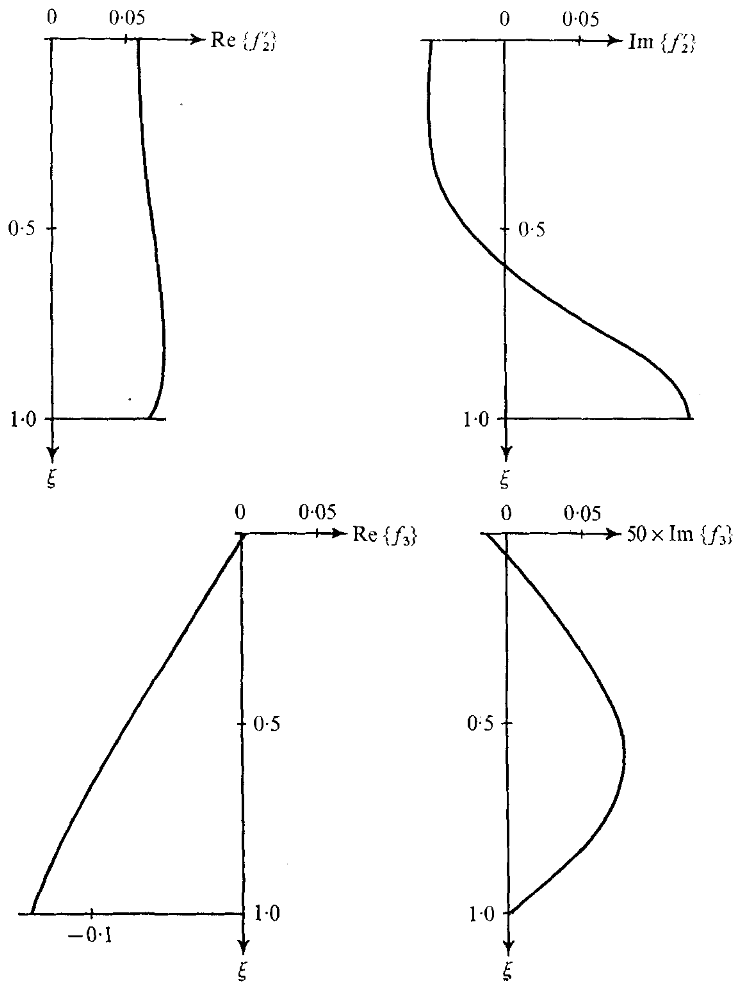

FIGURE 3. Typical example of the distribution of f_2' (lateral velocity component, see (10*c*)) and of f_3 (vertical velocity components, see (10*d*)) renormalized by division by h_0/D. Note that the lateral velocity component has opposite signs in the upper and lower part of a vertical (indicating helical flow) and that it is very much larger than the vertical component. The curves correspond to $k_1 D = 0{\cdot}14$, $k_2 D = 0{\cdot}15$, $V_0/U_{f0} = 16$ and a Froude number $F_0 = 0{\cdot}20$.

the Gauss–Legendre quadrature formulae, which approximate the integral of a function by a weighted sum of function values at the interior collocation points. In the practical calculations it is easier not to use (14) directly. From (11), (12*a*) and (12*b*) we get an alternative formula

$$\frac{i\epsilon}{k_1 D^2 U_{f0}}(f_2')'' + \frac{14\cos\beta\xi}{\beta^2}f_2' = \frac{-k_2 D}{(k_1 D)^2 + (k_2 D)^2}\left\{\frac{14\sin\beta\xi}{\beta}f_3 + \frac{14\cos\beta\xi}{\beta^2}f_3' + \frac{i\epsilon}{U_{f0}k_1 D^2}f_3'''\right\}. \quad (34)$$

Figure 3 shows a typical example of f_2' and f_3 renormalized by division by h_0/D.

5. The stability criterion

So far, we have been concerned with the problem of steady flow over a fixed bed. Strictly speaking, this flow pattern is not adequate for investigation of the development of the meandering, but here it is important to note that the migration velocity of the macroscopic features of the river bed, such as shoaling and meandering, is so extremely small that its neglect will lead to no detectable error at all.

The standard factor $\exp[ik_1x_1]$ used so far to describe the double periodicity should consequently be replaced by

$$\exp[ik_1(x_1-at)] = E,$$

in which $a = a_r+ia_i$ is the complex migration velocity of the sand waves. For example, we must have

$$h = h_0(\cos k_2x_2)\,E.$$

The continuity equation for the sediment motion is

$$\frac{\partial q_{s1}}{\partial x_1}+\frac{\partial q_{s2}}{\partial x_2} = -(1-n)\frac{\partial h}{\partial t}, \tag{35}$$

in which q_s denotes the rate of transport of sediment (volume of sediment grains per second per unit width) and n is the porosity of the sand.

Now we need a relation between q_s and the hydraulic parameters. From Engelund's similarity hypothesis (see Engelund 1967) we have

$$f\Phi = 0{\cdot}1\theta^{\frac{5}{2}}, \tag{36}$$

where f is the friction factor defined as

$$f = 2\tau_{13}/\rho V^2 \quad \text{for} \quad \xi = 1-h/D \tag{37}$$

and the non-dimensional sediment discharge is defined by

$$\Phi = q_{s1}/[(s-1)\,gd^3]^{\frac{1}{2}}. \tag{38}$$

For the transverse component q_{s2} we assume that

$$q_{s2} = q_{s0}\left[\frac{u_2}{U_{b0}}-c\frac{\partial h}{\partial x_2}\right]_{\xi=1-h/D}, \tag{39}$$

where c is a non-dimensional positive proportionality factor and q_{s0} denotes the transport rate for the basic flow. When $\partial h/\partial x_2 > 0$, the grains are displaced downwards by gravitational forces, which reduce the transverse sediment transport, and vice versa.

After substitution into the continuity equation (35) we get, after some manipulation, the following expression for the complex migration velocity:

$$a = \frac{q_{s0}}{k_1D(1-n)}\{c_5-ic(k_2D)^2\}, \tag{40}$$

where $c_5 = k_1D\{-\frac{3}{2}-2\beta\cot\beta\}-2\beta k_2D\{f_2(1)-f_2(0)\}\cot\beta-2\beta f_3(1)\cot\beta$

$$+\frac{3\epsilon U_{b0}}{2DU_{f0}^2}\{k_2Df_2''(1)+f_3''(1)\}+k_2Df_2'(1).$$

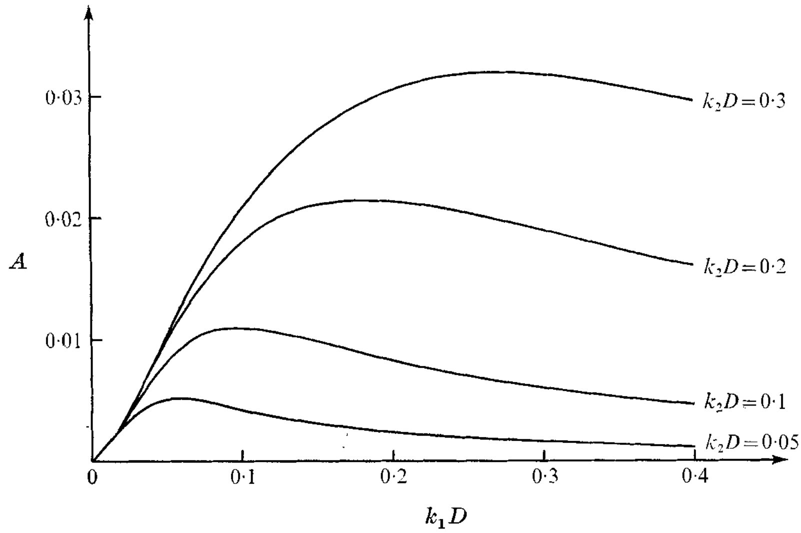

FIGURE 4. The amplification coefficient A as a function of the non-dimensional longitudinal wavenumber k_1D. The parameter is the non-dimensional lateral wavenumber k_2D. The curves correspond to $s = 2·65$, $D/d = 10000$ and $s_0 = 1·5 \times 10^{-4}$, i.e. $V_0/U_{f0} = 16$ and the Froude number $F_0 = 0·20$. The constant c in (40) is put equal to zero.

The flow is unstable if the imaginary part a_i of a is positive. The amplification factor is $\exp(a_i k_1)$, where

$$a_i k_1 = \frac{q_{s0}}{D(1-n)}\{\mathrm{Im}\{c_5\} - c(k_2D)^2\}. \tag{41}$$

The last factor in this expression will be called the amplification coefficient and will be denoted by A.

6. Discussion of results

Some numerical examples should be discussed in order to figure out the extent to which the mathematical model describes essential features known from experiments or from nature.

The amplification coefficient A was evaluated as a function of the channel width (characterized by k_2D) and the meander length (characterized by k_1D). A representative example is given in figure 4. The curves are sketched for $c = 0$. Any other value of c will only give each curve a vertical parallel displacement.

For all parameter combinations examined it was found that for some interval of k_1D the amplification factor assumed positive values, indicating instability. This is in accordance with the empirical fact that natural channels are found to meander or at least create more or less pronounced alternate shoals on apparently straight reaches.

An interesting feature is the peak of the curves, indicating that the amplification has a maximum. As was mentioned by Callander (1969), the greatest value of A for a given k_2D determines the wavelength of the developing meander. The theory seems to predict meander lengths of at least the right order of magnitude.

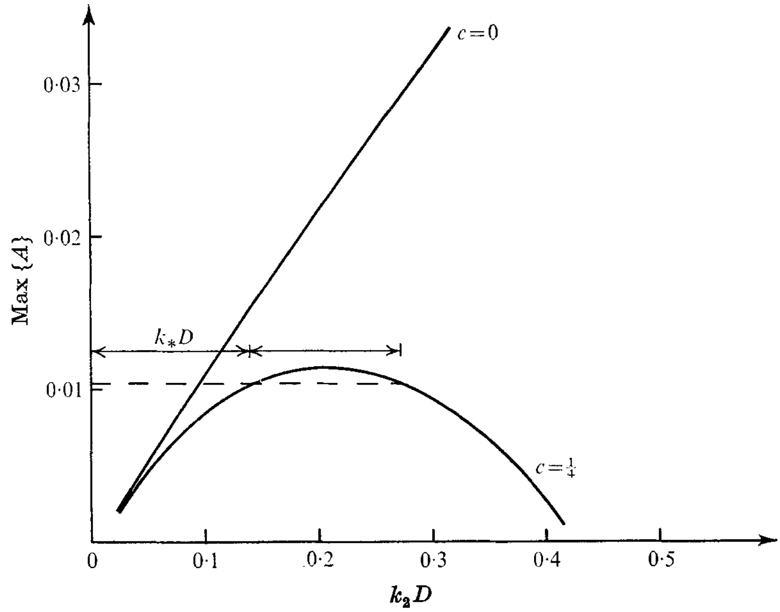

FIGURE 5. Max $\{A\}$ as a function of k_2D. The curves correspond to $s = 2{\cdot}65$, $D/d = 10000$ and $s_0 = 1{\cdot}5\times10^{-4}$, i.e. $V_0/U_{f0} = 16$ and the Froude number $F_0 = 0{\cdot}20$.

A striking feature is that the peak value max$\{A\}$ has maximum for a certain value of k_2D. This is illustrated in figure 5, where max$\{A\}$ is plotted against k_2D. The curve is sketched for $c = \frac{1}{4}$. Any other value of c will give a curve with a similar shape but with a different length scale. At present very little is known about the value of c, but preliminary experiments seem to indicate values of c smaller than unity. This fact has the interesting implication that it offers an explanation of the phenomenon of braiding, which will be accounted for shortly. Let us consider the particular abscissa k_*D for which the ordinate equals the ordinate corresponding to $2k_*D$, see figure 5.

In the situation thus defined the amplification is the same for a channel with half the actual width, or – put in another way – we get the same amplification for $m = 1$ and $m = 2$, the number m being defined by (16).

For $k_2 > k_*$ we shall always find the greatest amplification for $m = 1$. In physical terms this means that the river will meander. For k_2 slightly smaller than k_* we shall find the greatest amplification for $m = 2$, which means that the river is braiding. For still smaller values of k_2 the maximum amplification will occur for $m = 3$, 4 and so on.

Put in very simple words, the analysis indicates that for given hydraulic resistance and depth the river will exhibit meandering if the width is smaller than some threshold value B_*, while a wider river will braid in two or more courses – the more, the wider it is. This result seems to agree with the observations reported by Leopold & Wolman (1957).

A direct comparison between the mathematical model described in this paper and the many published data concerning quantities such as meander length, etc., is unfortunately not possible. Usually the field data originate from river meanders with a large (usually unspecified) sinuosity, while the present paper is concerned only with the initial stability of straight channels. The sinuosity that develops in nature because of the instability changes the hydraulic parameters (such as the slope) quite considerably.

Some flume experiments account carefully even for the initial part of the process, but are nevertheless unsuitable for comparison because the bed form is usually ripples rather than dunes. Little is known about the hydraulic resistance of ripple-covered beds, but it is certain that it differs considerably from that of a dune-covered bed.

7. Discussion of basic assumptions

In order to estimate the reliability of the mathematical model suggested above, a brief discussion of the basic assumptions will be needed. This will reveal some possibilities for future improvements.

Compared with most field conditions the present model gives an unrealistic treatment of the side banks. Even for the case of fixed banks the description is not quite realistic, because the particular secondary currents discussed by Shen & Komura (1968) have not been taken into account.

Another important point is the application of a simplified velocity profile in the basic flow and the use of the eddy-viscosity concept. The latter is motivated by several successful examples in engineering hydraulics.

The hydraulic roughness of a duned bed and its variation with flow conditions is a subject on which our knowledge is very uncertain and approximate. This will, however, mostly affect the numerical magnitude of our result, but hardly the principle.

Another point may be of importance, as far as the numerical predictions are concerned. This is the question of the influence of a certain part of the sediment load being carried in suspension rather than as a bed load, as was assumed here. As the formative discharge is usually rather high, we must in practice expect that a certain fraction of the total load is suspended. This will introduce a phase shift between the tractive shear and the total sediment load not accounted for in the present model.

REFERENCES

Ackers, P. & Charlton, F. G. 1970 The slope and resistance of meandering channels. *Proc. Inst. Civ. Engrs Paper*, no. 7362, 349–370.

Blench, T. 1966 *Mobile-Bed Fluviology*. University of Alberta, Edmonton, Canada.

Callander, R. A. 1969 Instability and river channels. *J. Fluid Mech.* **36**, 465–480.

Einstein, H. A. & Shen, H. W. 1964 A study of meandering in straight alluvial channels. *J. Geophys. Res.* **69**, 5239–5247.

Engelund, F. 1967 Hydraulic resistance of alluvial streams. Discussion, *J. Hyd. Div. A.S.C.E.* **93**, no. HY4, 287–296.

Engelund, F. 1970 Instability of erodible beds. *J. Fluid Mech.* **42**, 225–244.

Engelund, F. & Fredsøe, J. 1971 Three-dimensional stability analysis of open channel flow over erodible bed. *Nordic Hydrology*, **2**, 93–108.

Engelund, F. & Hansen, E. 1967 *A Monograph on Sediment Transport in Alluvial Streams*. Copenhagen: Danish Technical Press.

Langbein, W. B. & Leopold, L. B. 1966 River meanders – theory of minimum variance. *U.S. Geological Survey, Professional Paper*, 422-H.

Leopold, L. B. & Wolman, M. G. 1957 River channel patterns: braided, meandering and straight. *U.S. Geological Survey, Professional Paper*, 282-B.

Reynolds, A. J. 1965 Waves on the erodible bed of an open channel. *J. Fluid Mech.* **22**, 113–133.

Shen, H. W. & Komura, S. 1968 Meandering tendencies in straight alluvial channels. *J. Hyd. Div. A.S.C.E.* **94**, no. HY4, 997–1016.

Villadsen, J. V. & Stewart, W. E. 1967 Solution of boundary-value problems by orthogonal collocation. *Chem. Engng Sci.* **22**, 1483–1501.